Problem Set 2

Handed Out: Tuesday 2015/02/03
Due: Tuesday 2015/02/10 (before class)

1. **Operator Mathematics**. Prove the following useful properties of the commutator and the trace:
   (a) $\mathrm{Tr}(A\,B) = \mathrm{Tr}(B\,A)$.
   (b) $\mathrm{Tr}(A\,B\,C) = \mathrm{Tr}(C\,A\,B) = \mathrm{Tr}(C\,A\,B)$ (the trace of a product of operators is conserved if the operators are cyclicly permuted).
   (c) $[A\,B, C] = A\,[B, C] + [A, C]\,B$
   (d) $[A, B\,C] = [A, B]\,C + B\,[A, C]$
   (e) $\mathrm{Tr}(A\,[A, B]) = 0$ (A is trace-orthogonal to $[A, B]$).

2. **Operator Rotations**. Derive closed-form analytical expressions for $\sigma(t)$ in the following cases. Rotations (a-c) show up when considering the time evolution of a quantum harmonic oscillator; rotation (d) appears in the calculation of spin magnetization evolving under off-resonance irradiation.
   (a) $\sigma(t) = e^{-i\omega t I_z}\, I_+\, e^{+i\omega t I_z}$.
   (b) $\sigma(t) = e^{-i\omega t (a^\dagger a + 1/2)}\, a^\dagger\, e^{+i\omega t (a^\dagger a + 1/2)}$ and $\sigma(t) = e^{-i\omega t (a^\dagger a + 1/2)}\, a\, e^{+i\omega t (a^\dagger a + 1/2)}$.
   (c) $\sigma(t) = e^{-i\omega t (a^\dagger a + 1/2)}\, x\, e^{+i\omega t (a^\dagger a + 1/2)}$.
   (d) $\sigma(t) = e^{-i\mathcal{H}t}\, I_z\, e^{+i\mathcal{H}t}$ where $\mathcal{H} = \Delta\omega I_z + \omega_1 I_x$. Justify your answer with a picture and examine the limiting case where $\Delta\omega \to 0$.

3. **Properties of Spin Angular Momentum Operators**.
   (a) Supply the unitary operator U required to carry out the following rotations
      i. $U^\dagger I_z U = I_x$
      ii. $U^\dagger I_z U = \dfrac{1}{\sqrt{2}}(I_x + I_z)$
      iii. $U^\dagger I_z U = \dfrac{1}{\sqrt{3}}(I_x + I_y + I_z)$
   (b) Show $\mathrm{Tr}(I_x^2) = \mathrm{Tr}(I_y^2) = \mathrm{Tr}(I_z^2)$.
   (c) Consider the angular momentum operators for a spin $I = 1/2$ particle, I_x, I_y, and I_z. Show that $e^{i\theta I_z}$ can be expanded as follows

$$e^{i\theta I_z} = c_1(\theta)\,\mathbf{1} + c_2(\theta)\,I_z \tag{1}$$

   where $\mathbf{1}$ is the 2×2 identity matrix. Determine $c_1(\theta)$ and $c_2(\theta)$. Hint: Taylor series.
   (d) We would like to derive an expansion for $e^{i\theta I_z}$ for a spin $I = 1$ particle. To do so, it is helpful to know the Löwdin projection operator theorem. The first part of the theorem states that any function f of an operator A can be expanded as

$$f(A) = \sum_j P_j f(\lambda_j) \tag{2}$$

   where λ_j is jth eigenvalue of A, P_j is the projection operator onto the jth eigenvector of A, and the sum is over all the eigenvalues of A. The second part of the theorem is the finding that the projection operator can be written in terms of the operator and its eigenvalues, as

$$P_j = \prod_{i \neq j} \frac{(A - \lambda_i)}{(\lambda_j - \lambda_i)} \tag{3}$$

   where you should note that the product is over all i not equal to j.
      i. Use the Löwdin projection operator theorem to derive Eq. 1 and determine $c_1(\theta)$ and $c_2(\theta)$.
      ii. Use the theorem to derive an expression analogous to Eq. 1 but valid for a spin $I = 1$ particle.

4. **Curie-Law Magnetization**. Use the density matrix to derive the Curie Law, which predicts that the bulk magnetization of N spin-I particles is

$$M = \frac{N\gamma^2\hbar^2 I(I+1)}{3k_B T} B_0 \tag{4}$$

   where γ is the particles' gyromagnetic ratio, k_B is Boltzmann's constant, T is temperature, and B_0 is the applied magnetic field. State any approximations that you had to make to derive Eq 4.

5. **Pulsed Irradiation of Nuclear Magnetization**. A pulsed, circularly-polarized radiofrequency magnetic field of duration 5μs flips proton magnetization from the $+z$ to the $-z$ axis. What is the strength, B_1, of the radiofrequency field (in mT)?