Problem Set 1

Handed Out: Thursday 2015/01/22
Due: Tuesday 2015/02/03 (before class)

1. **Quantum and Classical Harmonic Oscillator**. In this problem we use the density operator to calculate the uncertainty in the position of an harmonic-oscillator wavefunction at thermal equilibrium and we compare the result to the result of the analogous classical-mechanical calculation.

   (a) Model the oscillator classically using the Hamiltonian $\mathcal{H} = p^2/2m + kx^2/2$ where p, m, k, and x are oscillator momentum, mass, spring constant, and position, respectively. Calculate $\langle x^2 \rangle_{\text{cl}}$ using classical statistical mechanics.

   (b) Write down the quantum mechanical density matrix assuming that the oscillator is in thermal equilibrium with a bath at temperature T and is governed by the Hamiltonian $\mathcal{H} = \hbar\omega_0(a^\dagger a + 1/2)$. Here $\hbar = 6.626 \times 10^{-34}$ J s is Planck's constant, ω_0 is the oscillator's resonance frequency, and $a^\dagger$ and a are the harmonic oscillator raising and lowering operators. Calculate $\langle x^2 \rangle_{\text{qm}}$ using this density matrix.

   (c) Compare the classical and quantum mechanical results for the root-mean-square variation in oscillator position; examine high T and low T limits. The low-temperature limit of $\langle x^2 \rangle_{\text{qm}}$ is called the standard quantum limit (sql).

   (d) Consider a harmonic oscillator with mass $m = 1.67 \times 10^{-27}$ kg and spring constant $k = 100$N/m. These values are roughly appropriate for a carbon-hydrogen bond. Take the oscillator to be at a temperature of $T = 300$ K and calculate $x_{\text{rms}}^{\text{cl.}} = \sqrt{\langle x^2 \rangle_{\text{cl}}}$. Compare it to $x_{\text{rms}}^{\text{sql}}$.

2. **Spin Density Operator**. In this problem we analyze a molecular beam experiment designed to determine the gyromagnetic ratio of the electron. The experiment is sketched below in Fig. 1.

   The oven $\mathcal{A}$ expels atomic hydrogen. Consider that each atom has a total electron spin $S = 1/2$; we will neglect the nuclear spin of the nucleus in our analysis. The atom passes through a Stern-Gerlach filter $\mathcal{B}$ that selects the spin-up projection along the y axis ($|+\rangle_y$ in Cohen-Tannoudji's notation (see C-T, Chapter IV, page 395). The atom then passes through a region of homogenous magnetic field $\mathcal{C}$. The field is directed along the x axis, has magnitude B_0, and acts on the spin for a distance L. The atom travels through a second Stern-Gerlach filter $\mathcal{D}$ that selects the spin-up projection along the z axis, $|+\rangle_z$. Finally, the atom is passed on to the detector $\mathcal{E}$.

   (a) Assume that all the atoms emitted from the oven $\mathcal{A}$ have the same velocity v_x. The atoms spend a time $T = L/v_x$ in the magnetic field. Write the spin part of the atom's density matrix just after exiting $\mathcal{B}$ (call it $\sigma(0)$), before exiting $\mathcal{C}$ (call it $\sigma(T)$), and after exiting $\mathcal{D}$. Write down an expression for the probability $\mathcal{P}_{|+\rangle_z}$ of detecting an atom at $\mathcal{E}$.

   (b) Take $L = 10^{-2}$ m and take the gyromagnetic ratio electron to be approximately $\gamma_e = 28.025\,\text{GHz}\,\text{T}^{-1}$. Plot the probability $\mathcal{P}_{|+\rangle_z}$ of detecting the atom at $\mathcal{E}$ as a function of magnetic field B_0. Rationalize the behavior of $\mathcal{P}_{|+\rangle_z}$.

   (c) Now model the oven $\mathcal{A}$ as emitting atomic hydrogen with a velocity distribution that follows Maxwell-Boltzmann statistics. That is, the probability of an atom being expelled with a velocity v_x is $\mathcal{P}(v_x) \propto e^{-mv_x^2/2k_bT}$ where $k_b = 1.38066 \times 10^{-23}$ J K^{-1} is Boltzmann's constant. Take the oven temperature to be 1000 K. Compute the average velocity $\langle v_x \rangle$ of hydrogen atoms emitted from the oven.

   (d) Suppose that a velocity filter is installed on the output of the oven. The filter passes particles which have velocities between $\langle v_x \rangle - \Delta v_x/2$ and $\langle v_x \rangle + \Delta v_x/2$. Write down integral expressions for density matrix $\sigma(T)$ and the probability $\mathcal{P}_{|+\rangle_z}$.

   (e) Suppose that the velocity filter is adjusted to pass 1) 10%, 2) 5%, and 3) 2% of the available beam flux. Compute numerically the range of velocities Δv_x passed in each of these cases. For each of these cases, plot the signal $\mathcal{P}_{|+\rangle_z}$ versus magnetic field B_0. This will require numerical integration. Make sure to sweep the magnetic field to a final value large enough to see at least fifty probability oscillations.

   (f) Which slit setting (10%, 5%, or 2%) allows for the most accurate determination of the gyromagnetic ratio? Why?

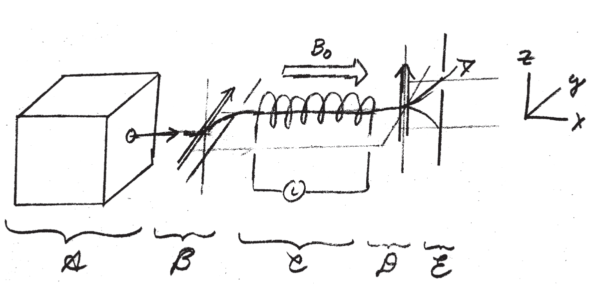

FIG. 1: Experiment to measure gyromagnetic ratio of the electron.